JOURNAL OF COMPUTATIONAL PHYSICS 31, 265–288 (1979)

Boundary Approximations and Accuracy in Viscous Flow Computations

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Received July 31, 1978

The way in which the boundary values of the vorticity are approximated in the numerical solution of the Navier-Stokes equations affects the rate of convergence and accuracy of the solutions. In this paper two classes of boundary approximations are studied. The problem of viscous flow in a square cavity is chosen as a model. Numerical solutions are obtained for Reynolds numbers 1, 10, 50, 100, 500 and 1000 and the iterative procedure is found to become faster with a decrease in the local accuracy of the boundary approximation. Detailed comparisons are carried out in order to determine accuracy of various numerical solutions. Several parameters, based on the numerical solutions, are found to vary monotonically and approach certain limiting values. These parameters are considered to be reliable indicators of accuracy and are recommended for comparison of numerical results obtained by different methods.

INTRODUCTION

Several papers have appeared in the last 15 years on the numerical solution of the Navier–Stokes equations governing the flow of a viscous incompressible fluid. The solution procedure consists of discretizing the differential equations and boundary conditions over the fluid flow region and solving the resulting system of algebraic equations. Finite difference methods are generally employed in the discretization although lately finite element methods have also been used. For two-dimensional and also for the axi-symmetric flows it is convenient to introduce stream function and vorticity as dependent variables. The equation of continuity is automatically satisfied and the resulting system consists of two coupled nonlinear equations which are solved numerically by some iterative procedure.

Aside from the fact that these coupled equations are nonlinear, there are several other difficulties associated with their solution. The major difficulty is that the values of vorticity on no-slip boundaries are not known a priori, while these values are needed in order to solve the discretized problem. Since there are no analytic solutions available for a physically interesting problem of viscous flow, it is difficult to ascertain the accuracy of the numerical solutions. The geometry of the problem also introduces additional difficulties in the numerical method. Some authors prefer the use of velocity-pressure formulation of the Navier–Stokes equations in order to avoid the difficulties arising from the introduction of vorticity. However, the pressure equation is complicated and introduces additional difficulties.

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0021-9991/79/050265–24$02.00/0

In order to test a numerical method, it is customary to choose a simple model problem. The flow of a viscous incompressible fluid in a rectangular cavity is often chosen for this purpose. Mathematically, this choice is rather unfortunate because of the corner singularities. Several numerical methods have been proposed in the literature. These methods differ in the choice of discretization schemes, the boundary approximations used to define vorticity on no-slip walls, and the methods used to solve the resulting systems of algebraic equations. The numerical solutions are usually compared in terms of the values of stream function or vorticity at some representative points and also by comparing the values of certain parameters of the flow.

It has now been established that central difference approximations provide convergent and accurate numerical solutions only for small values of the Reynolds numbers. For large Reynolds numbers, some upwind differencing is essential. However, the effect of various boundary approximations on the overall accuracy has not been studied systematically. From the study of the biharmonic equation, which is a special case of the Navier–Stokes equations with zero Reynolds number, it is known that the boundary approximations significantly affect the accuracy of the numerical solution as well as the rate of convergence of the overall procedure [7, 10, 12].

While most authors have used the conventional boundary approximation based upon a reflection principle, some others have considered higher order approximations. Many authors have encountered difficulties with second order boundary approximations. Due to these difficulties the second order formulas have been termed unstable and the conventional formula is generally recommended [20].

In this study we examine two classes of boundary approximation formulas. These formulas have been successfully used for the solution of the biharmonic equation [7, 10, 12]. In the class of first order formulas, the boundary vorticity is defined in terms of the given data and the stream function values at one point inside the flow region. For the second order formulas, the stream function values at two points inside the flow region are used to define the boundary vorticity. Effect of these boundary approximation formulas on the accuracy of the numerical solutions as well as on the rate of convergence of the numerical procedure is discussed.

In order to compare the accuracy of various numerical solutions, we consider several parameters such as the vortex centre, the maximum value of the stream function, the wall vorticity, the corner vortices and the shear force on the top wall of the cavity. Detailed comparisons are carried out for various Reynolds numbers and comments are made on the suitability of these parameetrs as reliable indicators of accuracy.

Numerical solutions of the cavity flow problem are obtained using a uniform mesh ($h = 0.05$) for Reynolds numbers 1, 10, 50, 100, 500 and 1000 with various boundary approximations. In addition, the second order boundary formula due to Woods [27] is also used for comparisons. Although most of the numerical solutions given here have been obtained by using upwind difference methods, some solutions at low Reynolds numbers have also been obtained using the central difference scheme.

It is concluded that if one desires to obtain a crude approximate solution of the Navier–Stokes equations, it can be obtained very cheaply using an appropriate boundary approximation formula. We also discuss why some second order formulas are considered unstable and indicate how a suitable choice of boundary formulas can yield a numerical solution which is comparable in accuracy but easier on computing budgets.

2. The Cavity Flow Problem

The flow of a viscous fluid in a square cavity is governed by the following Navier–Stokes equations:

$$\Delta\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega \tag{2.1}$$

$$L\omega \equiv \Delta\omega + R\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right) = 0. \tag{2.2}$$

The boundary conditions are given by

$$\begin{aligned}
&\psi = 0, \quad \frac{\partial\psi}{\partial x} = 0, \quad \text{when} \quad x = 0 \text{ or } 1,\\
&\psi = 0, \quad \frac{\partial\psi}{\partial y} = 0, \quad \text{when} \quad y = 0,\\
&\psi = 0, \quad \frac{\partial\psi}{\partial y} = -1, \quad \text{when} \quad y = 1.
\end{aligned} \tag{2.3}$$

In order to solve these equations numerically, the square cavity is covered by a uniform mesh of width h. The set of mesh points is defined by

$$D_h = \{(x_i, y_j): x_i = ih, y_j = jh; i, j = 1, 2, \ldots, n-1; nh = 1\}.$$

The set of boundary mesh points is denoted by ∂D_h. The finite difference approximation of the stream function equation is given by

$$\Delta_h\psi_{ij} = h^{-2}[\psi_{i,j+1} + \psi_{i,j-1} + \psi_{i-1,j} + \psi_{i+1,j} - 4\psi_{ij}] = \omega_{ij}, \quad 1 \leqslant i, j \leqslant n-1 \tag{2.4}$$

The upwind scheme for the vorticity equation (2.2) is given by

$$\begin{aligned}
L_h\omega_{ij} \equiv K_1\omega_{i+1,j} + K_2\omega_{i,j+1} + K_3\omega_{i-1,j} + K_4\omega_{i,j-1}&\\
- K_0\omega_{i,j} = 0, \qquad 1 \leqslant i, j \leqslant n-1,&
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&K_1 = 1 - \tfrac{1}{4}R(\beta_{ij} - |\beta_{ij}|), \qquad K_3 = 1 + \tfrac{1}{4}R(\beta_{ij} + |\beta_{ij}|)\\
&K_2 = 1 + \tfrac{1}{4}R(\alpha_{ij} + |\alpha_{ij}|), \qquad K_4 = 1 - \tfrac{1}{4}R(\alpha_{ij} - |\alpha_{ij}|),\\
&K_0 = 4 + \tfrac{1}{2}R(|\alpha_{ij}| + |\beta_{ij}|);
\end{aligned}$$

and

$$\alpha_{ij} = \psi_{i+1,j} - \psi_{i-1,j}, \qquad \beta_{ij} = \psi_{i,j+1} - \psi_{i,j-1}.$$

The boundary values of ψ and ω must be prescribed in order to solve the systems of equations (2.4), (2.5). The boundary values of ψ are

$$\psi_{ij} = 0, \qquad (x_i, y_j) \in \partial D_h, \tag{2.6}$$

whereas the boundary values of ω are defined in terms of ψ ($\omega \equiv -\Delta\psi$) and may be prescribed as

$$\omega_{ij} = -\Delta_h \psi_{ij}, \qquad (x_i, y_j) \in \partial D_h. \tag{2.7}$$

Assuming (x_i, y_j) lies on the left boundary $x = 0$, then

$$\omega_{0j} = -h^{-2}[\psi_{1j} + \psi_{-1,j} + \psi_{0,j+1} + \psi_{0,j-1} - 4\psi_{0,j}]. \tag{2.8}$$

The value of $\psi_{-1,j}$ is undefined in the above expression as the point (x_{-1}, y_j) lies outside the square cavity. Since the normal derivative $\partial u/\partial x$ is known on the boundary, the following reflection formula may be used to define $\psi_{-1,j}$:

$$(\psi_x)_{0,j} = \frac{\psi_{1j} - \psi_{-1,j}}{2h} + O(h^2)$$

or

$$\psi_{-1,j} = \psi_{1,j} - 2h(\psi_x)_{0,j} + O(h^3). \tag{2.9}$$

Substitution of (2.9) into (2.8) yields the following "conventional approximation":

$$\omega_{0j} = h^{-2}[2\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - 4\psi_{0,j} - 2h(\psi_x)_{0,j}]. \tag{2.10}$$

Similar expressions may be written for the other parts of the boundary. The approximation (2.10) has been widely used in the litearture [4, 14, 15, 17, 20, 21, 24, 26].

Some authors [9, 16, 20, 24] divide the set D_h of interior mesh points into two sets $D_{h,1}$ and $D_{h,2}$, where $D_{h,1}$ contains mesh points adjacent to the boundary (at a distance h) and $D_{h,2} = D_h - D_{h,1}$, and solve different algebraic problems on the two sets of mehs points. As an example, Greenspan [9] solved the stream function equation (2.4) only on $D_{h,2}$ (i.e., for $2 \leqslant i, j \leqslant n - 2$) and defined the ψ-values on $D_{h.1}$ using interpolation such as

$$\psi_{1j} = \tfrac{1}{4}[\psi_{2j} + 3\psi_{0,j} + 2h(\psi_x)_{0,j}]. \tag{2.11}$$

It is noted that the boundary value of ω_{0j} depends on ψ_{1j} from (2.10); the value of ψ_{1j} depends upon ψ_{2j} from (2.11). The combined effect is that ω_{0j} is defined by the following formula

$$\omega_{0j} = -h^{-2}[\tfrac{1}{2}\psi_{2j} - -\psi_{0j} + \psi_{0,j+1} + \psi_{0,j-1} - h(\psi_x)_{0,j}]. \tag{2.12}$$

Observe that the approximation for ω_{0j} given by (2.10) depends upon the value of $\psi_{1,j}$ and the known data. Similarly, the approximation (2.12) depends upon the value of $\psi_{2,j}$. In general it is possible to define a first order approximation to ω_{0j} in terms of ψ_{pj} and the known data by a formula called the $(p, 0)$ formula given by

$$\omega_{0j} = -h^{-2}\left[\frac{2}{p^2}\psi_{p,j} + \psi_{0,j+1} + \psi_{0,j-1} - (2 + 2p^{-2})\,\psi_{0,j} - \frac{2}{p}h(\psi_x)_{0,j}\right] \quad (2.13)$$

where p is a positive integer. Orszag and Israeli [18] have mentioned this formula as a possible generalization of Thom's formula for a one-dimensional model problem.

A second order approximation, denoted symbolically as the (p, q) formula, is given by

$$\begin{aligned}\omega_{0j} = -h^{-2}\Big[&-2q^3\alpha\psi_{p,j} + 2p^3\alpha\psi_{q,j} + \psi_{0,j+1} + \psi_{0,j-1}\\ &- 2\left(\frac{p+q}{pq}\right)h(\psi_x)_{0,j} - (2 + 2p^3\alpha - 2q^3\alpha)\,\psi_{0,j}\Big],\end{aligned} \quad (2.14)$$

where p, q are positive integers, $p \neq q$, and $\alpha = p^{-2}q^{-2}(p-q)^{-1}$. The (p, q) formula defines the values of $\omega_{0,j}$ in terms of two ψ-values at (x_p, y_j) and (x_q, y_j), besides the known data. In particular, the (2, 1) formula is given by

$$\omega_{0,j} = -h^{-2}[-\tfrac{1}{2}\psi_{2,j} + 4\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - 3h(\psi_x)_{0,j} - \tfrac{11}{2}\psi_{0,j}]. \quad (2.15)$$

This formula has been used by several authors. Roache [21] calls it Jensen's formula. It has also been called Briley's formula. Wu [28] recommends the use of second order boundary approximations but also notes that with such formulas the principle of total vorticity conservation may be violated. Several authors (see [21]) have called the formula (2.15) unstable. In fact the rate of convergence of the iterative scheme is slow when (2.15) is used. We discuss this matter in Section 5.

Some other second order formulas from the class (2.14) are:

The (3, 1) formula:

$$\omega_{0,j} = -h^{-2}[-\tfrac{1}{9}\psi_{3,j} + 3\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - \tfrac{8}{3}h(\psi_x)_{0,j} - \tfrac{44}{9}\psi_{0,j}]; \quad (2.16)$$

The (3, 2) formula:

$$\omega_{0,j} = -h^{-2}[-\tfrac{4}{9}\psi_{3,j} + \tfrac{3}{2}\psi_{2,j} + \psi_{0,j+1} + \psi_{0,j-1} - \tfrac{5}{3}h(\psi_x)_{0,j} - \tfrac{47}{18}\psi_{0,j}]; \quad (2.17)$$

The (4, 3) formula:

$$\omega_{0,j} = -h^{-2}[-\tfrac{1}{2}\psi_{4,j} + \tfrac{8}{9}\psi_{3,j} + \psi_{0,j+1} + \psi_{0,j-1} - \tfrac{7}{6}(\psi_x)_{0,j} - \tfrac{43}{18}\psi_{0,j}]. \quad (2.18)$$

Orszag and Israeli have mentioned some (2, 1) and (3, 1) formulas for their one-dimensional model problem [18, Eqns. 38, 39]. Howevei, their formulas seem to be of first order in general.

The boundary approximations given by the class of formulas (2.13) and (2.14) include most of the formulas used by other authors. An exception is the so called Woods formula given by

$$\omega_{0,j} = -h^{-2}[3\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - 5\psi_{0,j} - 3h(\psi_x)_{0,j}] + \tfrac{1}{2}\omega_{1,j}\,. \qquad (2.19)$$

This formula has been used by several authors [1, 2, 3, 21, 23, 27]. It defines the boundary vorticity in terms of $\psi_{1,j}$ and $\omega_{1,j}$ besides the known data. This formula has a truncation error of order h^2 and one expects to obtain more accurate results using Woods' formula compared to those obtained using the conventional (1, 0) formula or any other first order formula of the class (2.13). Since the accuracy of Woods' formula is of the same order as those of formulas of the class (2.14), we expect the solutions to be comparable. In order to make our discussion complete, we also give results obtained using Woods' formula. Intuitively the formula (2.19) may appear better than the class of formulas (2.14) because of the inclusion of additional information. However, this is not the case as will be shown here.

3. Iterative Procedure

The solution of the discrete Navier–Stokes equations (2.4) and (2.5) is obtained by the following iterative procedure:

(a) Start with some initial approximation $\omega^{(m)}$ of the vorticity with $m = 0$. If no such approximation is available, set $\omega^{(m)} \equiv 0$.

(b) Solve the stream function equation (2.4) to obtain $\psi^{(m+1)}$ from:

$$\Delta_h \psi_{ij}^{(m+1)} = -\omega_{ij}^{(m)}. \qquad (3.1)$$

(c) Determine the boundary values of the vorticity from the formula (2.13) or (2.14). Call these values $\bar{\omega}^{(m+1)}$. Obtain the modified boundary values $\omega^{(m+1)}$ using a smoothing (or damping) parameter δ:

$$\omega^{(m+1)} = (1 - \delta)\,\bar{\omega}^{(m+1)} + \delta\omega^{(m)}, \qquad 0 < \delta < 1. \qquad (3.2)$$

(d) Solve the vorticity equation (2.5) to obtain $\omega^{(m+1)}$ from

$$L_h \omega_{ij}^{(m+1)} = 0. \qquad (3.3)$$

(e) Repeat the steps (b) to (d) for $m = 1, 2, \ldots$ until some convergence criterion is met.

The iterative steps (b) to (d) form an "outer iteration." If the equations (3.1) or (3.3) are solved by an iterative procedure, then the steps (b), (d) are called inner iterations for ψ, ω.

4. Computational Preliminaries

In general it is easy to solve the discrete Poisson equation (3.1) by using the successive overrelaxation method or some other iterative procedure provided the region under consideration is rectangular and some optimal relaxation parameters are available a priori. This, however, is not possible for the equation (3.2). Direct solvers are now available which are quite efficient and use of these solvers eliminates the search for optimal relaxation parameters. Since our basic interest is to test various boundary approximations, we chose to eliminate the uncertainty arising from the inner iterations in order to make an objective comparison of our final results. Although many of the direct methods are computationally as fast as the iterative methods, the storage requirements become prohibitive when h is small.

A nonzero value of the damping parameter δ in (3.2) is essential for the convergence of the numerical procedure (see [7, 10] for proof of the case $R = 0$). For an estimate of δ, we first determine the growth factor ρ of the outer iterations. The value of ρ is estimated by using $\delta = 0$ for a small number of iterations in the procedure of Section 3 and computing

$$\rho \approx \frac{\| \psi^{(n+1)} - \psi^{(n)} \|}{\| \psi^{(n)} - \psi^{(n-1)} \|}, \qquad n \text{ large}. \tag{4.1}$$

The norm used in (4.1) is the maximum norm $\| \psi \| = \max | \psi_{ij} |$. Let $\mu = (\rho - 1)/(\rho + 1)$. For convergence of the outer iterations, δ should be chosen [7] such that

$$\mu \leqslant \delta < 1, \tag{4.2}$$

and a near optimal value of δ is given by

$$\delta_{\text{opt}} = \frac{\rho}{\rho + 2}. \tag{4.3}$$

Most of our computations were started with $\omega^{(0)} \equiv 0$, although this was not necessary. The outer iterations were stopped when

$$\| \omega^{(n)} - \omega^{(n-1)} \| < \epsilon. \tag{4.4}$$

This convergence criterion also guarantees

$$\| \psi^{(n)} - \psi^{(n-1)} \| < \epsilon. \tag{4.5}$$

The value of ϵ was chosen to be 10^{-4}.

5. Numerical Results

We have obtained numerical solutions for $R = 1, 10, 50, 100, 500$ and 1000 with a mesh size $h = 1/20$ using various boundary approximations. Some results for $h = 1/10$ are available from a previous study [11]. Solutions have also been obtained by using the central difference scheme for low Reynolds numbers ($R = 1, 10, 50, 100$).

We found that with a fixed boundary approximation the growth factors ρ of the outer iterations remained virtually constant for the range of Reynolds numbers considered by us. In Table 1, we give representative values of ρ, μ and δ_{opt} defined in Eqs. (4.1)–(4.3). It is noted that values of ρ, μ and δ decrease with the increasing values of p when the $(p, 0)$ formula (2.13) is used and with increasing values of p, q when the (p, q) formula (2.14) is used to define the boundary vorticity. The effect of decreasing δ_{opt} is to increase the rate of convergence of the outer iterations. Thus, one could expect a faster convergence when boundary formulas with larger values of p and q are used.

TABLE I

Growth Factors, Stability Range, and Optimum Parameters

Boundary approximation p, q[a]	Growth factor ρ	Lower bound for smoothing parameter μ	Optimum smoothing parameter δ
1, 0	10.6	0.8276	0.84
2, 0	4.86	0.6587	0.71
3, 0	2.98	0.4975	0.60
4, 0	2.07	0.3485	0.51
2, 1	16.25	0.8841	0.89
3, 2	8.6	0.7917	0.81
4, 3	5.75	0.7037	0.74
5, 4	4.23	0.6176	0.68
6, 3	4.80	0.6552	0.71
7, 2	6.43	0.7308	0.76
Woods[b]	10.8	0.8305	0.85

[a] $q = 0$ indicates the first order boundary approximation (2.13), $q \neq 0$ indicates the second order boundary approximation (2.14).

[b] Woods formula given by Eq. (2.19).

In Table 2, we give the values of δ actually used and the number of iterations required for convergence. Clearly, the number of iterations go down, sometimes even drastically, with an increase in value of p or q. It is anticipated that with larger

TABLE II

Damping Parameters δ and Number of Iterations N

Boundary approximation	δ_{opt} from	Reynolds number R							
		10		50		100		500	
p, q	Table I	δ	N	δ	N	δ	N	δ	N
1, 0	0.84	0.85	46	0.85	50	0.85	63	0.86	137
2, 0	0.71	0.695	43	0.66	36	0.695	31^a	0.70	73
3, 0	0.60	0.58	16^a	0.58	18	0.58	19^a	0.60	51
4, 0	0.51	0.50	12^a	0.50	15	0.50	16^a	—	
2, 1	0.89	0.905	69	0.89	71	0.89	89	0.91	202
3, 2	0.81	0.825	43	0.81	42	0.83	60	0.84	133
4, 3	0.74	0.73	51	0.73	30	0.73	34^a	0.75	94
5, 4	0.68	—		0.68	20^a	0.68	26^a	—	
Woods	0.85	0.85	40^a	0.85	45^a	0.85	66^c	0.85	124^a

[a] The initial values of $\omega^{(0)}$ were not taken to be zero. Thus these numbers are not comparable with the other data, but are presented for the sake of completeness.

values of p, q the boundary approximations and hence the numerical solutions become progressively less accurate. This is borne out by our experience with the biharmonic equation and confirms the obvious that less accurate results are cheaper to obtain.

It is not possible to make statements about the accuracy of the numerical solutions in absolute terms, because no analytical solutions of the problem are available. On the other hand, from our experience with the biharmonic equation it is possible for us to select certain representative values of the solution and also certain parameters which can provide a good indication of the relative accuracies of the numerical solutions.

(i) *Qualitative Comparisons*

The stream function and vorticity profiles obtained with different boundary approximations are all qualitatively correct and compare well with the profiles published elsewhere. As an example, the stream function and vorticity profiles for $R = 50$, $h = 1/20$ obtained with six different boundary approximations of the type (2.13), (2.14) all look alike [11].

If only qualitative results are needed, it is advisable to use larger values of p and q in the boundary approximations (2.13), (2.14). These formulas are very economical compared to the conventional method. We do not, however, recommend the use of extremely large values of p and q in obtaining these solutions. As in the case of the biharmonic equation [7, 10, 12], we believe that the second order boundary formulas could be used with p, q as large as desired without seriously affecting the accuracy provided that $pq = O(1/h)$.

(ii) *Quantitative Comparisons*

The numerical procedure under consideration has previously been applied to the biharmonic equation in a square under a variety of boundary conditions [7, 12]. Since exact solutions of the biharmonic equation are known in most cases, it is possible to make specific comments about the accuracy of various numerical solutions. In particular, the overall accuracy of the numerical solutions of the biharmonic equation increases with an increase in the accuarcy of the boundary approximations. Here overall accuracy is measured in terms of the maximum error at the mesh points. The second order formulas (2.14) yield more accurate results than the first order formulas (2.13), at least for moderate values of p and q. For the mesh sizes of the order of 1/20 and 1/25 one could not, in general, expect a pointwise accuracy $\geqslant 10^{-4}$ in ψ and $\geqslant 10^{-2}$ in ω when the convergence criteria (4.4), (4.5) are used with $\epsilon = 10^{-4}$. We believe that a similar trend exists for the Navier–Stokes equations, at least for moderate Reynolds numbers. To expect any better accuracy in terms of the values of stream function and vorticity seems pointless to us unless both the mesh size is reduced and the convergence criterion is modified. With this in mind, we now examine several parameters which have been quoted in the literature to compare various numerical solutions.

(a) *Maximum Value of ψ, the Vortex Centre and Vorticity at Vortex Centre*

The point at which the value of ψ attains its absolute maximum is called the centre of the primary vortex (vc). We denote the coordinates of this point by $(\bar{x}, \bar{y})$ and values of ψ, ω at the vortex centre by ψ_{vc} and ω_{vc}. The values of these parameters are presented in Table 3 for various R, p and q. It may be noted that the location of the vortex centres given here are limited by the mesh size used in these calculations and the actual vortex centre may lie anywhere in the square $(\bar{x} \pm h, \bar{y} \pm h)$. The results in Table 3 clearly show that $(\bar{x}, \bar{y})$ is virtually independent of the boundary approximation and hence is an unreliable parameter to compare the accuracy of various numerical solutions.

The variations in the values of ψ_{vc} as given in Table 3 are not very large at small Reynolds numbers. However, these variations increase with the increase in the Reynolds number. Moreover, the values of ψ_{vc} vary monotonically with the accuracy of the boundary approximations and ψ_{vc} appears to be a reliable indicator of accuracy.

It may be noted that since $(\bar{x}, \bar{y})$ does not represent the true centre of the primary vortex (resolution errors of order h in each direction), the values of ψ at $(\bar{x}, \bar{y})$ also may not be the true values of ψ_{max} in the cavity. In comparing the results obtained using different mesh sizes, one must allow for this variation and it might be more appropriate to compare the values of ψ at a fixed point which may be in the vicinity of the vortex centre.

TABLE III

Vortex Center, Maximum Stream Function, and Vorticity at the Vortex Center.

Reynolds number R	Boundary approximation p, q	Vortex center $(\bar{x}, \bar{y})$	Upwind scheme ψ_{max}	Upwind scheme ω_{vc}	Central scheme ψ_{max}	Central scheme ω_{vc}	Comparable results from the literature	
1	1, 0	(0.5, 0.75)	0.0993	3.00	0.0993	3.00		
	2, 0	(0.5, 0.75)	0.0982	2.95	0.0982	2.95	$\psi_{max}=0.0995$, $h=1/20$	[14]
	3, 0	(0.5, 0.75)	0.0955	2.86	0.0955	2.84		
	4, 0	(0.5, 0.75)	0.0918	2.70	0.0917	2.70		
	2, 1	(0.5, 0.75)	0.0995	3.02	0.0995	3.02		
	3, 2	(0.5, 0.75)	0.1000	3.03	0.1000	3.03		
	4, 3	(0.5, 0.75)	0.1006	3.05	0.1005	3.05		
	5, 4	(0.5, 0.75)	0.1008	3.05	0.1008	3.05		
	Woods	(0.5, 0.75)	0.1082	3.33				
50	1, 0	(0.45, 0.75)	0.1000	3.00	0.0981	3.17		
	2, 0	(0.4, 0.75)	0.0979	3.10	0.0959	3.05	$\psi_{max}=0.101$, $h=1/40$	[9]
	3, 0	(0.4, 0.75)	0.0942	2.90	0.0921	2.84		
	4, 0	(0.4, 0.75)	0.0897	2.69	0.0877	2.63		
	2, 1	(0.45, 0.75)	0.1006	3.03	0.0987	3.21		
	3, 2	(0.45, 0.75)	0.1011	3.05	0.0991	3.23		
	4, 3	(0.4, 0.75)	0.1009	3.25	0.0990	3.21		
	5, 4	(0.4, 0.75)	0.1002	3.20	0.0981	3.15		
	Woods	(0.45, 0.75)	0.1101	3.40				
100	1, 0	(0.40, 0.75)	0.0985	3.05	0.0953	3.28		
	2, 0	(0.35, 0.75)	0.0945	3.18	0.0914	3.02	$\psi_{max}=0.1032$, $h=1/50$	[3]
	3, 0	(0.35, 0.75)	0.0901	2.87	0.0868	2.68	$\psi_{max}=0.1015$, $h=1/20$	[14]
	4, 0	(0.35, 0.75)	0.0854	2.59	0.0805	2.39	$\psi_{max}=0.0955$, $h=1/20$	
							$=0.1022$, $h=1/50$	[4]
	2, 1	(0.4, 0.75)	0.1000	3.13	0.0971	3.36	$\omega_{vc}=3.136$, $h=1/20$	
	3, 2	(0.4, 0.75)	0.0997	3.13	0.0965	3.37	$=3.145$, $h=1/50$	
	4, 3	(0.4, 0.75)	0.0979	3.05	0.0949	3.28	$\psi_{max}=0.1026$, $h=1/50$	[16]
	5, 4	(0.35, 0.75)	0.0959	3.28	0.0928	3.11	$\omega_{vc}=3.155$, $h=1/50$	
	Woods	(0.4, 0.75)	0.1095	3.57			$\psi_{max}=$ 0.1053, $h=1/15$	
							0.1043, $h=1/29$	[23]
							0.1040, extrapolated	

Table continued

TABLE III—*Continued*

Reynolds number R	Boundary approximation p, q	Vortex Center $(\bar{x}, \bar{y})$	Upwind scheme ψ_{max}	Upwind scheme ω_{vc}	Comparable results from the literature	
500	1, 0	(0.3, 0.75)	0.0721	2.93		
	2, 3	(0.3, 0.75)	0.0687	2.00	$\psi_{max} = 0.105, h = 1/20$	[9]
	3, 0	(0.3, 0.65)	0.0693	1.69		
	2, 1	(0.3, 0.75)	0.0791	2.57		
	3, 2	(0.3, 0.8)	0.0707	2.92		
	4, 3	(0.3, 0.75)	0.0680	2.85		
	Woods	(0.3, 0.8)	0.0799	3.73		
1000	3, 0	(0.3, 0.75)	0.0541	1.76	$\psi_{max} = 0.0812, h = 1/50$	[3]
	2, 1	(0.3, 0.8)	0.0599	2.63	$= 0.0691, h = 1/30$	
	3, 2	(0.25, 0.8)	0.0520	3.31	$\psi_{max} = 0.0971, h = 1/20$	[14]
	Woods	(0.25, 0.85)	0.0587	4.13		

The values of vorticity at the vortex centre are more sensitive to the boundary approximations. It may be noted from Table 3 that the ω_{vc} values decrease with the decrease in the accuracy of the first order boundary approximation (2.12) and they [illegible] mation (2.14). The true values of ω_{vc} probably lie between the values obtained with the (1, 0) and the (2, 1) formulas. It must be noted that the values of ω may not be accurate beyond at most one or two decimal places for the mesh size used here and to read in any more in the results might be misleading. We believe that ω_{vc} should be used with caution in comparison of various numerical solutions. In some sense it indirectly implies comparing the values of the stream function i.e., ψ_{vc}.

(b) *Some Vorticity Values on the Boundaries*

Some authors quote the value of the vorticity at the midpoint (0.5, 1) of the moving wall. The values of vorticity on the boundary are obtained from extrapolation formulas involving the values of ψ at neighboring mesh points. In effect this is a process of numerical differentiation and the results strongly depend upon the formula used and the mesh size. The values of ψ near the boundary are small and erorrs in these values are amplified by a factor h^{-2}. Thus a comparison of various numerical solutions on the basis of vorticity values at one boundary mesh point does not make sense.

In Table 4, we give the values of $\omega(0.5, 1.0)$ for several values of R, p and q. It is noted that the ω values generally decrease when p is increased in the class of first order formulas. On the other hand, these values of ω increase with increasing values of p or q

TABLE IV

Vorticity at the Midpoint of the Moving Wall

Reynolds number R	Boundary approximation p, q	Vorticity value at $x = \frac{1}{2}, y = 1$		Comparable results from the literature
		Upwind scheme	Central scheme	
1	1, 0	5.87	5.88	
	2, 0	5.85	4.86	
	3, 0	5.75	5.75	
	4, 0	5.52	5.52	
	2, 1	5.88	5.89	
	3, 2	5.88	5.89	
	4, 3	5.93	5.94	
	5, 4	6.04	6.04	
	Woods	6.39		
100	1, 0	6.68	7.79	7.1376, 15 × 15 spline
	2, 0	7.25	8.06	6.6876, 29 × 29 spline
	3, 0	6.95	7.42	6.5376, extrapolated spline
	4, 0	6.33	6.59	6.2970, 19 × 19 spline with unequal spacing
	2, 1	6.28	7.44	[23]
	3, 2	6.82	8.07	8.916, 15 × 15 finite difference
	4, 3	7.48	8.60	6.696, 57 × 57 finite difference
	5, 4	7.73	8.57	6.548, extrapolated finite difference [26]
	Woods	6.55		
1000	2, 0	13.77		21.508, 17 × 17 finite difference
	2, 1	20.45		16.198 65 × 65 finite difference
	3, 2	19.55		14.254 17 × 17 spline [22]
	Woods	26.58		

in the class of second order formulas. The values of $\omega(0.5, 1)$ however have the same order of magnitude irrespective of the boundary approximation.

From the published results of Rubin and Graves [23] for $R = 100$, it is noted that that the values of $\omega(0.5, 1)$ reduces as the mesh is refined. The value of ω obtained with extrapolated spline is 6.5376 which "improves" to 6.2970 when a spline with unequal spacing is used. This seems to give the impression that a smaller value of $\omega(0.5, 1)$ indicates improved accuracy of the overall solution. If this were to be accepted, then our solution obtained with the (2, 1) formula is even more accurate than that obtained with unequally spaced splines [23]. Moreover, our solutions with the (1, 0), (2, 1) and Woods' formulas would seem to be more accurate than the solution obtained with a 57×57 mesh [26] where $\omega(0.5, 1)$ is 6.696.

TABLE V

Extrapolated Vorticity Values Obtained from Numerical Solutions with (2, 1) Boundary Approximation

Boundary approximation p, q	Vorticity values at $x = \frac{1}{2}, y = 1$	
	$R = 50$	$R = 100$
1, 0	5.77	6.23
2, 0	5.69	6.19
3, 0	5.45	5.82
4, 0	5.16	5.39
5, 0	4.85	4.99
8, 0	4.00	4.02
2, 1	5.84	6.28
3, 1	5.93	6.44
3, 2	6.18	6.92
4, 1	5.97	6.51
4, 2	6.23	6.98
4, 3	6.34	7.10
5, 1	6.00	6.54
5, 2	6.26	6.99
5, 3	6.36	7.07
5, 4	6.39	7.01
Woods	11.50	12.48

In order to illustrate this point further, we have taken the converged solutions for $R = 50$ and 100 obtained with the "most accurate" (2, 1) boundary approximation under consideration and used this solution to extraploate the values of $\omega(0.5, 1)$ using equations (2.13) and (2.14) with various values of p and q. From Table 5 it is clear that with appropriate choice of p and q, one could obtain very low or very high values of ω. However, most of these values are equal in an order of magnitude comparison. It is concluded that the values of vorticity at any boundary point is an unreliable parameter to compare numerical solutions.

TABLE VI

Size of Upstream and Downstream Corner Vortices

Reynolds number R	Boundary approximation p, q	Size of the corner vortex: Upstream Y_u	Size of the corner vortex: Downstream Y_d	Comparable data from the literature	
10	1, 0	0.071	0.051		
	2, 0	[a]	[a]	$Y_u = 0.09$, $h = 1/50$	[3]
	3, 0	[a]	[a]	$Y_u = 0.06$, $h = 1/40$	[4]
	4, 0	[a]	[a]		
	2, 1	0.086	0.079		
	3, 2	0.071	0.066		
	4, 3	0.056	0.053		
	Woods	0.119	0.113		
100	1, 0	0.126	0.064	$Y_u = 0.15$, $Y_d = 0.12$, $h = 1/50$	(3)
	2, 0	0.093	[a]		
	3, 0	[a]	[a]		
	4, 0	[a]	[a]	$Y_u = 0.15$, $Y_d = 0.08$, $h = 1/40$	[4]
	2, 1	0.136	0.073	$Y_d = 0.06$	[16]
	3, 2	0.138	0.061		
	4, 3	0.136	0.051		
	5, 4	0.108	[a]		
	Woods	0.203	0.108		
				$Y_u = 0.33$	[3]
1000	2, 0	0.328	0.283	$Y_u = 0.3$ } $Y_d = 0.15$ }	[16]
	2, 1	0.564	0.435		
	3, 2	0.554	0.557	$Y_u = 0.28$	[19, 25]
	Woods	0.644[b]	0.607[b]		

[a] No corner vortex resolved in this case.
[b] Tertiary flow observed in this case.

(c) *Sizes of Corner Vortices*

Small counterrotating vortices indicating backflow have been observed, both experimentally and numerically, in the bottom corners of the square cavity. We were able to detect such corner vortices for Reynolds numbers as low as 1, at least with the more accurate boundary approximations. As the accuracy of the boundary approximations reduce, the size of the corner vortices also goes down. This may be taken as a measure of accuracy of the numerical solutions, but the accuracy would only be qualitative. The reason is that the size of the corner vortices is measured by the coordinates of the boundary mesh points where the vorticity is zero. In view of our comments about the vorticity values on the boundary this parameter too cannot be treated as a reliable parameter to compare the accuracy.

In Table 6, we present the values Y_u and Y_d of the vertical distances, measured from the bottom of the cavity, of the separation points (where $\omega = 0$) along the upstream and downstream walls. It is noted that the values of Y_u , Y_d using Woods'

TABLE VII

Stream Function Values $\psi(\zeta)$ near the Singular Upstream Corner[a]

Reynolds number R	Boundary approximation p, q	Values of $\psi(\zeta)$ $\zeta = 0.05$	$\zeta = 0.2$	Comparable results[b] from the literature
50	1, 0	0.0177	0.0716	
	2, 0	0.0161	0.0720	$\psi(0.05) \in (0.01, 0.04)$ $h = 1/40$ [15]
	3, 0	0.0145	0.0708	$\psi(0.2) \sim 0.07$
	4, 0	0.0132	0.0683	
	2, 1	0.0184	0.0713	
	3, 2	0.0181	0.0720	
	4, 3	0.0174	0.0730	
	5, 4	0.0167	0.0739	
	Woods	0.0231	0.0773	
500	1, 0	0.0239	0.0667	$\psi(0.05) \in (0.01, 0.03)$ $h = 1/20$ [9]
	2, 0	0.0204	0.0637	$\psi(0.02) \sim 0.07$
	3, 0	0.0171	0.0590	
	2, 1	0.0251	0.0693	
	3, 2	0.0240	0.0658	
	4, 3	0.0224	0.0648	
	Woods	0.0307	0.0710	

[a] ζ is the nondimensional distance along the diagonal.

[b] The data obtained from the graphs published in [9, 15].

formula are larger than any other values in the same class. We also noticed tertiary flow at $R = 1000$ using Woods' formula. This phenomena was noticed with other boundary approximations before convergence but it disappeared in the converged solutions.

(d) *Values of ψ near the Singular Upstream Corner*

Comparison of ψ values at some interior mesh points which are not too close to the boundary does provide an indication of accuracy, as discussed earlier. Similarly the values of ω can be compared although this in effect results in a comparison of the values of a linear combination of ψ-values at five mesh points, amplified by h^{-2}. The values of ψ near the boundaries are very small and a meaningful comparison of these values may require a much greater precision. As noted previously, the values of ω on or near the boundary are susceptible to the choice of differentiation formula used as well as the mesh size. In addition, slight inaccuracies in ψ-values get amplified to larger amounts.

O'Brien [17] has suggested that a more critical test of a numerical solution can be made near the upstream corner singularity, because if anything is to go wrong it would probably happen there. This argument seems fallacious because the numerical results are incorrect near the singularities due to the averaging processes normally employed in the calculations. The values of ψ near the singular corners are very small and, in our opinion, provide only an order of magnitude information. In Table 7, we present some values of ψ at the diagonal passing through the singular upstream corner. The distance along the diagonal is given as the fraction of the total length of the diagonal.

(e) *Total Shear Force*

The total shear stress on the moving wall per unit depth is given by

$$F = \int_0^a \mu \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\text{wall}} d\bar{x} \tag{5.1}$$

which can be nondimensionalized as $\bar{F} = FR/u_0^2 a\rho$. Then $\bar{F}$ is given by

$$\bar{F} = \frac{R}{u_0^2 a\rho} \int_0^a \mu \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\text{wall}} d\bar{x} = \int_0^1 \left(\frac{\partial u}{\partial y}\right)_{y=1} dx = \int_0^1 (\omega)_{y=1}\, dx. \tag{5.2}$$

Clearly, $\bar{F}$ is the average velocity gradient, or the average vorticity, on the moving wall. Although the values of ω on the wall are obtained by numerical differentiation [illegible] In Table 8, we present the values of $\bar{F}$ for various values of R, p and q. The integration is carried out using the trapezoidal rule and the value of ω at the singular corners, which do not enter our numerical computations, have been taken to be zero.

An examination of Table 8, reveals that the value of $\bar{F}$ consistently decreases with a decrease in the accuracy of the boundary approximations. The central difference schemes also produce slightly larger values of $\bar{F}$ which might be taken as an indicator of a slightly better accuracy of the central difference schemes. However, the values of $\bar{F}$ obtained with Woods' formula are substantially larger than any of the other values. In general the Woods' formula overestimates the values of vorticity and hence $\bar{F}$ as will be discussed later.

TABLE VIII

Total Shear Force on the Moving Wall

Reynolds number R	Boundary approximation p, q	Total shear force $\bar{F}$		Comparable results from the literature	
		Upwind scheme	Central scheme		
1	1, 0	11.20	11.21		
	2, 0	9.06	9.07		
	3, 0	7.67	7.68		
	4, 0	6.65	6.65	9.677, $h = 1/20$	[14]
	2, 1	12.71	12.72		
	3, 2	11.22	11.23		
	4, 3	10.14	10.15		
	5, 4	9.29	9.29		
	Woods	14.87			
10	1, 0	11.14	11.23		
	2, 0	9.04	9.09		
	3, 0	7.66	7.69		
	4, 0	6.65	6.66		
	2, 1	12.63	12.75		
	3, 2	11.16	11.25		
	4, 3	10.10	10.17		
	Woods	14.75			
50	1, 0	11.29	11.75		
	2, 0	9.21	9.47		
	3, 0	7.81	7.96		
	4, 0	6.75	6.84		
	2, 1	12.73	13.33		
	3, 2	11.28	11.77		
	4, 3	10.25	10.62		
	5, 4	9.42	9.68		
	Woods	14.85			

Table continued

TABLE VIII—*Continued*

Reynolds number R	Boundary approximation p, q	Total shear force $\bar{F}$		Comparable results from the literature
		Upwind scheme	Central scheme	
100	1, 0	11.89	12.94	
	2, 0	9.71	10.27	10.192, $h = 1/20$ [14]
	3, 0	8.15	8.44	11.39,[a] $h = 1/15$
	4, 0	6.95	7.11	17.71,[a] 19 × 19 unequally spaced grid
	2, 1	13.34	14.76	
	3, 2	11.89	12.98	
	4, 3	10.82	11.59	
	5, 4	9.90	10.41	
	Woods	15.65		
500	1, 0	16.70		
	2, 0	12.51		
	3, 0	9.63		
	2, 1	19.04		
	3, 2	16.72		
	4, 3	14.38		
	Woods	23.07		
1000	2, 0	13.94		13.361, $h = 1/20$ [14]
	2, 1	24.00		
	3, 2	19.90		
	Woods	29.16		

[a] Computed from data given in [26].

(f) *Vorticity Conservation Laws*

It is easy to derive the following conservation laws for the square cavity (see [8]):

$$\bar{\omega} = \iint_D \omega \, dx \, dy = 1 \tag{5.3}$$

and

$$\oint_{\partial D} \frac{\partial \omega}{\partial n} \, ds = 0. \tag{5.4}$$

Once the numerical solutions have been obtained the value of $\bar{\omega}$ can be computed numerically. In Table 9, we give the values of $\bar{\omega}$ obtained by using the trapezoidal

TABLE IX

Vorticity Conservation $\bar{\omega} = \iint \omega \, dx \, dy$

Reynolds number R	Boundary approximation p, q	$\bar{\omega}$ for vorticity conservation: Upwind scheme	Central scheme	Reynolds number R	Boundary approximation p, q	$\bar{\omega}$ for vorticity conservation: Upwind scheme	Central scheme
1	1, 0	0.9501	0.9500	100	1, 0	0.9500	0.9501
	2, 0	0.9259	0.9257		2, 0	0.9271	0.9120
	3, 0	0.8970	0.8967		3, 0	0.8900	0.8694
	4, 0	0.8663	0.8659		4, 0	0.8523	0.8309
	2, 1	0.9597	0.9598		2, 1	0.9528	0.9673
	3, 2	0.9559	0.9559		3, 2	0.9608	0.9632
	4, 3	0.9484	0.9483		4, 3	0.9553	0.9465
	5, 4	0.9393	0.9391		5, 4	0.9411	0.9243
	Woods	1.0476			Woods	1.0492	
10	1, 0	0.9501	0.9501	500	1, 0	0.9501	
	2, 0	0.9277	0.9255				
	3, 0	0.8993	0.8962		3, 0	0.8025	
	4, 0	0.8687	0.8654		2, 1	0.9891	
	2, 1	0.9581	0.9599		3, 2	0.9643	
	3, 2	0.9559	0.9560				
	4, 3	0.9496	0.9483				
	Woods	1.0450			Woods	1.1521	
50	1, 0	0.9500	0.9501	1000	2, 0	0.8051	
	2, 0	0.9309	0.9215		2, 1	1.0388	
	3, 0	0.9005	0.8877		3, 2	0.9519	
	4, 0	0.8672	0.8535				
	2, 1	0.9539	0.9619				
	3, 2	0.9573	0.9580		Woods	1.2259	
	4, 3	0.9538	0.9484				
	5, 4	0.9452	0.9355				
	Woods	1.0417					

rule. It is noted that the values of $\bar{\omega}$ are closer to unity when a more accurate boundary approximation is used. It is concluded that $\bar{\omega}$ is a reliable parameter for comparing or ascertaining the accuracy of numerical solutions.

The second conservation law involves the values of $\partial\omega/\partial n$ on the boundary mesh points. Since the vorticity values on the boundary are unreliable, determination of the normal derivative would be further inaccurate and the use of the second conservation law does not appear very promising. Wu [28] has, however, noted that with the first order approximations, in particular the (1, 0) formula the condition (5.4) would be exactly satisfied whereas with the second order approximations, specifically the (2, 1) formula, this condition may be violated.

It may be noted from Table 9 that while almost all values of $\bar{\omega}$ obtained with the (p, q) formulas are smaller than unity, those obtained with Woods' formula are all larger than 1. This is an indication that the errors of the numerical solutions obtained with Woods' formula lie on the other side of the true solutions. This also explains why the values of ψ_{max} and ψ_{vc} obtained with Woods' formula (Table 3) are higher than all other values. Such phenomena is also noticed in the spline solutions of Rubin and Khosla [22] and Rubin and Graves [23] where the values of ψ_{max} ($R = 100$) go down as a finer mesh is taken (see also Table 3). Similar comments apply to the sizes of corner vortices (Table 6), stream function values near the upstream singular corner (Table 7) and total shear force on the moving wall (Table 8). In addition, the values of $\bar{\omega}$ for $R = 500$ and 1000 clearly indicate the inaccuracies of the solutions obtained with the Woods' formula, at least for large Reynolds numbers, compared to those obtained with many of the first and second order formulas of the type (2.13), (2.14).

It is noted that the values of $\bar{\omega}$ for the conventional (1, 0) boundary formula are exactly those predicted theoretically, $\bar{\omega} = 1 - h + O(h^2)$.

(g) *Velocity Profiles*

Many authors [5, 6, 13, 18, 20–22, 25] have compared certain velocity profiles, usually along some mid-section or a diagonal. Also the velocities do represent the physical phenomena more clearly than the numerical values of ψ and ω. Calculation of velocity values also involves numerical differentiation of one order lower than the vorticity. In some sense the velocity profiles do provide a satisfactory method of comparing numerical solutions. It is, however, difficult to measure accuracy using these profiles.

6. Conclusions

We have examined a numerical procedure for solving two-dimensional Navier–Stokes equations on a model problem of a square cavity. The values of vorticity on the no-slip boundaries are approximated using various extrapolation formulas of first and second order. It is found that most of these boundary approximations, for moderate values of p and q, yield numerical solutions which display the expected characteristics of the fluid flow. At low Reynolds numbers, these solutions are even quantitatively comparable. In general, the two point formulas ($q \neq 0$) give more accurate results than the one point formulas ($q = 0$). In particular, the (2, 1) formula, which is also known as Jensen's formula [21], gives more accurate results than any of the other formulas examined here. This formula is also the most expensive in terms of the number of iterations required for convergence and we recommend the use of (3, 2) or (4, 3) formulas to obtain slightly less accurate solutions with substantially reduced cost.

Several authors have considered Jensen's formula and Woods' formula and found them to be unstable (see, e.g., [21]). However, these formulas are not unstable. They do require a substantial amount of damping of the boundary values of vorticity and hence are quite slow in convergence. As seen in Section 5, it is quite easy to obtain near optimal values of the damping parameter which may be used to get the best possible rate of convergence. In general it was found that the more accurate the boundary approximation in terms of the truncation error, the larger the cost of obtaining the numerical solutions.

With larger Reynolds numbers and a finer mesh, it is probable that the (2, 1) formula would require a very large amount of damping (even larger than 99 %). Such damping would make the convergence extremely slow and one may look at other approximations which are of the same order but cost much less.

We have examined a number of parameters which are often quoted in the literature to compare various numerical solutions. A systematic study of these parameters has been carried out in order to determine whether they are reliable indicators of accuracy. While most of the parameters examined here give some qualitative idea of accuracy, we have isolated the following parameters which are believed to provide a quantitative distinction between various numerical solutions: maximum value of stream function; total shear force on the moving wall of the cavity; and the total vorticity $\bar{\omega}$ defined in Eq. (5.3). The value of vorticity at the vortex centre can also be compared but this in essense means a comparison of a linear combination of ψ-values in the neighborhood of the vortex centre, amplified by h^{-2}.

Most of our discussion of Section 5 is based on our experiences with the biharmonic equation [7, 10, 12] which is a special case of the Navier–Stokes equations with $R = 0$. In this case, the accuracy of the numerical solution deteriorates when (p, q) boundary formulas are used with large p, q. Our conclusions on the suitability of certain parameters are also based on the trends found in other published results. As an example, the value of $\psi_{\max}$ generally increase with the accuracy of the boundary approximations of the type given here (Table 3). Similarly, the value of $\psi_{\max}$ increases with the refinement of the mesh in the finite difference methods [4]. On the other hand the value of $\psi_{\max}$ decreases with the refinement of the mesh in the spline methods [22, 23]. In all cases, there is a definite trend towards a limiting value of $\psi_{\max}$. From these observations one can safely conclude that the Woods' formula overestimates the $\psi_{\max}$ by about 10 %.

It has so far been believed that the central difference approximation of the vorticity transport equation (2.2) provides a higher order accuracy than the upwind difference scheme, at least for low Reynolds numbers. We did not find any perceptible difference between the solutions obtained with both types of difference schemes for $h = 1/20$ and $R = 1, 10, 50$ and 100. At higher Reynolds numbers the central difference scheme is either nonconvergent or gives inaccurate results.

Finally, when the same problem was solved using finite element methods under various boundary approximations of the type (2.13) and (2.14), it was found [13] that the rate of convergence of the iterative procedure was almost independent of the boundary approximation. In such a case it is advisable to use the (2, 1) formula.

Acknowledgments

The computations reported in this paper were carried out initially on UNIVAC 1308 at the Australian National University and subsequently on IBM370/158 at the University of Saskatchewan. The generous assistance of Drs. Bob Anderssen and Mike Osborne of the Australian National University is gratefully acknowledged. This research was supported by the National Research Council of Canada, the Papua New Guinea University of Technology and the University of Saskatchewan.

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